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A Class of Rough Surfaces and Their Fractal Dimensions

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Based on a function represented by the Cantor series and a binary fraction of real numbers, a class of rough surfaces is constructed in this paper. The method used completely differs from those previously developed. In addition, the fractal dimensions, as an indicator of the roughness of such surfaces, are also investigated. The calculating formulas for the box-counting and packing dimensions are derived, and the upper and lower bounds of the Hausdorff dimension are estimated.

Key Words: Cantor series; rough surface; fractal dimensions.

1. INTRODUCTION

Since Mandelbrot [1, 2] introduced his concepts, fractal geometry has provided a new and powerful framework in which to model and analyse a wide class of natural phenomena. How to construct fractal curves and surfaces (rough curves and surfaces) and analyse their complexity has become one of the most important topics in fractals. The traditional approaches to constructing those curves and surfaces mainly include geometrically recursive methods and infinite series representing ways. The resulting curves include the well-known Von Koch [3], Peano [4], and Weierstrass [5, 6] curves.

In recent years, several other techniques have been employed for constructing such curves and surfaces. One scheme has been to use wavelet

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decompositions(see, e.g., [7]), and another method that has been investigated a lot these past years is based on iterated function systems(IFS) [8]. Barnsley [9] has proved that, under certain conditions, it is possible to construct an IFS whose attractor is the graph of a nowhere differentiable continuous function called a fractal interpolation function (FIF). Massopust [10] has constructed a class of fractal interpolation surfaces by using IFS, which are the graphs of continuous functions from an oriented standard simplex $\sigma^2 \subset R^2$ into R. Now fractal interpolation surfaces have become a powerful tool for modeling many natural surfaces.

Research has shown that many natural rough surfaces, such as a topographic surface and a fracture surface in materials or rocks, exhibit both self-similar and self-affine fractal behaviors, or rather the self-affine fractal behavior. Most of the fractal characterization of some rough surfaces may be expressed by one sectional profile of those surfaces. Mandelbrot [2] suggested that the fractal dimension of a topographic surface can be obtained by adding 1.0 to the fractal dimension obtained from a single profile of that surface. From the viewpoint of theory, we should consider how to construct a fractal surface using a fractal profile. Usually, a special Cartesian product and a fractional Brownian motion surface may be used to approximate the natural fractal surface.

Generally speaking, a fractal surface is a special set in the three-dimension Euclidean space R^3, and usually it can be expressed as a bivariate function $z = f(x, y)$. Suppose A is a fractal profile in plane ZOX, $A = \{(x, z) : x \in [a, b], z = g(x)\}$, while B is an interval $[c, d]$ in R; then a simple fractal surface F can be described by Cartesian product

$$F = A \times B = \{(x, y, z) : x \in [a, b], y \in [c, d], z = g(x)\}.$$

This means the fractal surface F will be obtained by movement of the fractal profile A along the straight line containing B. In this case, the Hausdorff dimension of F equals one plus the Hausdorff dimension of A.

A fractional Brownian motion surface [11] is a completely stochastic fractal surface. It has been applied in computer graphics to generate various topographic surfaces.

However, most natural rough surfaces are more complicated than the special Cartesian product, but more regular than the fractional Brownian motion surface. Therefore the modeling of natural rough surfaces using the Cartesian product and Brownian motion may cause an obvious error. In this paper we will construct a class of rough surfaces based on a function represented by the Cantor series and a binary fraction of real numbers. We will also study the fractal dimensions of these surfaces. The constructed surfaces could be applied effectively to model many natural phenomena.

The paper is organized as follows. In Section 2, we briefly recall some basic definitions, notations, and related results on the box-counting, packing, and Hausdorff dimensions of a set, which will be used in our subsequent discussion. In Section 3, we construct a class of rough surfaces based on a previously defined function. In Section 4, we investigate the box-counting and packing dimensions of these surfaces and show that these two dimensions are the same. In Section 5, we give the upper and lower bound estimates for Hausdorff dimension.

2. PRELIMINARIES

In this section we briefly recall the definitions of various dimensions of a set useful for the following.

Let F be a bounded set of R^n and $N_\delta(F)$ be the smallest number of sets of diameter at most δ required to cover F. The upper and lower box-counting dimensions of F are then defined [11], respectively, by

$$\overline{\dim}_B F = \overline{\lim_{\delta\to 0}} \frac{\log N_\delta(F)}{-\log \delta}, \tag{1}$$

$$\underline{\dim}_B F = \underline{\lim_{\delta\to 0}} \frac{\log N_\delta(F)}{-\log \delta}. \tag{2}$$

If both limits are equal, then the common value, denoted by $\dim_B F$, is called the box-counting dimension of F.

There are some equivalent definitions of the box-counting dimension. In particular, we usually take $N_\delta(F)$ to be the number of δ-mesh cubes which intersect F in calculating the box-counting dimension of a graph of a function.

Let F be a nonempty set of R^n. For $s \geq 0$ and $\delta > 0$, define

$$H^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |F_i|^s : F \subset \bigcup_{i=1}^{\infty} F_i, 0 < |F_i| \leq \delta\right\},$$

where $|F_i|$ denotes the diameter of F_i. Since $H^s_\delta(F)$ increases as δ decreases, the following limit exists, but may be infinite:

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F).$$

$H^s(F)$ is called the s-dimensional Hausdorff outer measure of F. The Hausdorff dimension of F is then defined by [11]:

$$\dim_H F = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

Besides the Hausdorff and box-counting dimensions, the packing dimension of a set will be used also. Let F be a nonempty set of R^n;

$$P^s_\delta(F) = \sup\left\{\sum_{i=1}^{\infty} |B_i|^s\right\},$$

where $\{B_i\}_{i=1}^{\infty}$ is a collection of disjoint balls of radius at most δ with centers in F; and

$$P_0^s(F) = \lim_{\delta \to 0} P_\delta^s(F).$$

The s-dimensional packing measure is defined by

$$P^s(F) = \inf\left\{\sum_{i=1}^{\infty} P_0^s(F_i) : F \subset \bigcup_{i=1}^{\infty} F_i\right\}.$$

By means of $P^s(F)$, the packing dimension of the set F is defined by

$$\dim_P F = \sup\{s : P^s(F) = \infty\} = \inf\{s : P^s(F) = 0\}.$$

It is well known that

$$\dim_H F \leq \dim_P F \leq \overline{\dim}_B F.$$

As seen from the above definitions, the Hausdorff and packing dimensions of a set are defined in terms of measures and are extremely difficult to calculate directly. However, the following Lemma 1 provides an alternative definition of the packing dimension, which makes direct evaluation of the packing dimension possible.

LEMMA 1 [11]. *Let $F \subset R^n$ be a compact set. If for any open set V which intersects F, the identity*

$$\overline{\dim}_B(F \cap V) = \overline{\dim}_B F \tag{3}$$

holds, then $\dim_P F = \overline{\dim}_B F$.

In order to make a direct estimate for the Hausdorff dimension of a set, we introduce some notation and another lemma ([12]).

Notations. If W is a k-dimensional linear space over R, the triple $(W, \langle\cdot,\cdot\rangle, (e_1, \ldots, e_k))$ is called a Euclidean space with an orthonormal basis $(e_1, \ldots, e_k)$ and a scalar product $\langle\cdot,\cdot\rangle$. Sometimes for simplicity we will omit the symbols of the scalar product and the basis in the expression of such a triple.

We denote the Lebesgue measure on W by m_W.

A k-dimensional closed parallelepiped $D \subset W$ is said to be canonical if all its edges are parallel to vectors from the basis $(e_1, \ldots, e_k)$.

If $x \in W$ and $r > 0$, then $D(x, r)$ denotes the canonical cube with center at x and edges of length r. If the center is not specified we call such a cube an (r, W)-cube.

For $V \subset W$, a linear subspace of W, $\pi_V: W \to V$ denotes the orthogonal projection onto V.

LEMMA 2 [12]. *Let* $(W_1, (e_1, \ldots, e_m)), (W_2, (e_{m+1}, \ldots, e_{m+l}))$ *be two Euclidean spaces with bases and let* $W = W_1 \times W_2$. *Assume that for a Borel set* $K \subset W$ *the following two conditions are satisfied*:

(i) $m_{W_1}(\pi_{W_1}(K)) > 0$.

(ii) *There exist constants* $c_1, c_2 > 0$ *and* $0 < \alpha < 1$, *such that for every* (r, W_1)*-cube* D *with* $r > 0$ *sufficiently small and for every* $z \in W_2$ *there exists a* $(c_1 r, W_1)$*-cube* $D' \subset D$ *such that the parallelepiped* $D' \times D(z, c_2 r^\alpha)$ *is disjoint from* K.

Then $\dim_H K \geq C(\alpha, c_1, m) > \dim_H W_1 = m$, *where* $C(\alpha, c_1, m)$ *is a constant depending only on* α, c_1, *and* m.

Remark 1. It can be seen from the proof of Lemma 2 in [12] that

$$C(\alpha, c_1, m) = m + \frac{\alpha - 1}{\alpha} \cdot \frac{\log(1 - M^{-m})}{\log M} > m,$$

where $M = [2/c_1] + 1$ and $[x]$ denotes the integer part of x.

3. CONSTRUCTION OF A CLASS OF ROUGH SURFACES

We first introduce the concept of a Cantor series which will be used to construct a class of nowhere differentiable continuous functions in the remainder of this section.

Let $I = [0, 1]$ be the closed unit interval in R. Assume that $\{p_k\}$ is a given sequence of positive integers satisfying the conditions

$$p_k \geq 2, \qquad k = 1, 2, 3, \ldots.$$

We divide I into p_1 closed subintervals of equal length,

$$I_0 = \left[0, \frac{1}{p_1}\right], \quad I_1 = \left[\frac{1}{p_1}, \frac{2}{p_1}\right], \ldots, \; I_{p_1 - 1} = \left[\frac{p_1 - 1}{p_1}, 1\right].$$

These subintervals are called 1-order intervals. Generally, we divide every k-order intervals $I_{x_1 \cdots x_k} (x_i = 0, 1, \ldots, p_i - 1; i = 1, 2, \ldots, k)$ into p_{k+1} smaller subintervals of equal length

$$I_{x_1 \cdots x_k x_{k+1}} \qquad (x_{k+1} = 0, 1, \ldots, p_{k+1} - 1).$$

So, $(k+1)$-order intervals are obtained. Going on like this, we can obtain all k-order intervals $(k = 1, 2, 3, \ldots)$. For any sequence of integers $\{x_k\}$ satisfying the conditions

$$0 \leq x_k \leq p_k - 1, \qquad k = 1, 2, 3, \ldots, \tag{4}$$

by the nested interval theorem, $\bigcap_{k=1}^{\infty} I_{x_1\cdots x_k}$ then consists of a singleton in I. It is clear that for any $x \in I$ there exists a corresponding sequence of integers $\{x_k\}$ satisfying Eq. (4), such that

$$x = \bigcap_{k=1}^{\infty} I_{x_1\cdots x_k}. \tag{5}$$

From Eq. (5), it is easy to see that x can be expressed as

$$x = \sum_{k=1}^{\infty} \frac{x_k}{p_1 p_2 \cdots p_k}, \qquad 0 \leq x_k \leq p_k - 1. \tag{6}$$

Equation (6) is called the Cantor series expression of x, and x_k is said to be the kth digit of x.

If x is an endpoint of some subinterval and $x \neq 0, 1$, then there exists an index k_0 such that x can be expressed as

$$x = \sum_{k=1}^{k_0} \frac{x_k}{p_1 p_2 \cdots p_k} \tag{7}$$

and

$$x = \sum_{k=1}^{k_0-1} \frac{x_k}{p_1 p_2 \cdots p_k} + \frac{x_{k_0} - 1}{p_1 p_2 \cdots p_{k_0}} + \sum_{k=k_0+1}^{\infty} \frac{p_k - 1}{p_1 p_2 \cdots p_k}. \tag{8}$$

Only in this case there are two Cantor series expressions for x; however, the corresponding values of x are the same for these two expressions.

A Cantor series is a direct generalization of a fractional expression of real numbers. In particular, it is noted that when $p_k = b$ $(b \geq 2)$, Eq. (6) is just the b-adic expansion of x.

Now we proceed to construct a class of nowhere differentiable continuous functions $f(x)$ by means of the Cantor series expression of x.

Let $\{n_k\}, k = 0, 1, 2, \ldots,$ be a sequence of integers which is strictly increasing, and satisfy $n_0 = 0$. For any $x \in I$, with reference to the Cantor series expression (6), we define a function $f(x)$ by

$$u = f(x) = 0 \cdot u_1 u_2 \cdots u_k \cdots = \sum_{k=1}^{\infty} \frac{u_k}{2^k}, \qquad u_k \in \{0, 1\}, \tag{9}$$

where $u_1 = 0$. When $k > 1$, u_k are defined as follows:

(i) If $x_i(n_{k-2} < i \leq n_k)$ are all 0 or $x_i = p_i - 1$, then $u_k = u_{k-1}$.

(ii) If $x_i(n_{k-1} < i \leq n_k)$ are all 0 (or $x_i = p_i - 1$), but there exists at least a $x_{j_0}(n_{k-2} < j_0 \leq n_{k-1})$ such that $x_{j_0} \neq 0$ (or $x_{j_0} \neq p_{j_0} - 1$), then $u_k = 1 - u_{k-1}$.

(iii) If $x_i(n_{k-1} < i \leq n_k)$ are not simultaneously 0 and are also not simultaneously $p_i - 1$, then $u_k = 0$.

It is clear that u_k as defined above depends only on the former n_k digits of the Cantor series expression of x. And it is easy to verify that, although there exist two Cantor series expressions for some x, the corresponding function value $f(x)$ is uniquely determined. Hence, for convenience, we make a contract, in what follows, that if x has two Cantor series expressions we then always take the form (7) instead of the form (8). With this convention, there exists only one Cantor series expression for every $x \in I$.

In [13] it has been shown that the above function $f(x)$ is continuous on the interval I and that under one of the following so-called nondifferentiability conditions,

(A) $p_k \geq 3, k = 1, 2, \ldots$;

(B) $p_k \geq 2, k = 1, 2, \ldots$, and $\lim_{k\to\infty}(n_k - k) = \infty$, the function $f(x)$ has nowhere a finite derivative in the interval (0,1).

Obviously, when the sequences $\{p_k\}$ and $\{n_k\}$ take different forms, we can obtain different nowhere differentiable continuous functions.

Next we construct a class of bivariate functions on I^2 based on the above function $f(x)$.

For any $(x, y) \in I^2$, suppose x and y can be expressed as the Cantor series

$$x = \sum_{k=1}^{\infty} \frac{x_k}{p_1 p_2 \cdots p_k}, \qquad 0 \leq x_k \leq p_k - 1, \tag{10}$$

and

$$y = \sum_{k=1}^{\infty} \frac{y_k}{p_1 p_2 \cdots p_k}, \qquad 0 \leq y_k \leq p_k - 1, \tag{11}$$

respectively.

Define

$$z = F(x, y) = \lambda(x) f(x) + \mu(y) f(y) + e, \tag{12}$$

where $\lambda(x), \mu(y)$ are two nonzero continuously differentiable functions on the interval I satisfying $\lambda_1 = \min\{|\lambda(x)| : 0 \leq x \leq 1\} > 0$ and $\mu_1 = \min\{|\mu(y)| : 0 \leq y \leq 1\} > 0$, e is a constant, and $f(x)$ is the function defined by Eq. (9).

It is clear that the bivariate functions constructed in Eq. (12) are continuous and it is easy to verify that they are nowhere differentiable

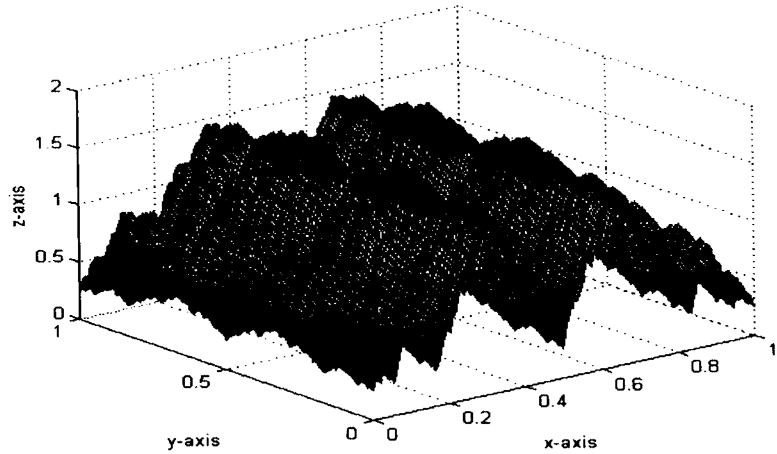

FIG. 1. Graph of $z = \lambda f(x) + \mu f(y) + e$ with $\lambda = 0.5$, $\mu = 2.5$, $e = 0.25$, $p_k = 3$, and $n_k = k$.

in I^2, provided that one of the nondifferentiability conditions (A) and (B) holds.

Since the graphs of such functions, surfaces, have in general noninteger fractal dimensions known from the following sections, they may display very complicated structures. Hence, such surfaces are a class of rough surfaces. Figures 1–5 show such rough surfaces in several typical cases. In the following sections, we will discuss the fractal properties of these surfaces.

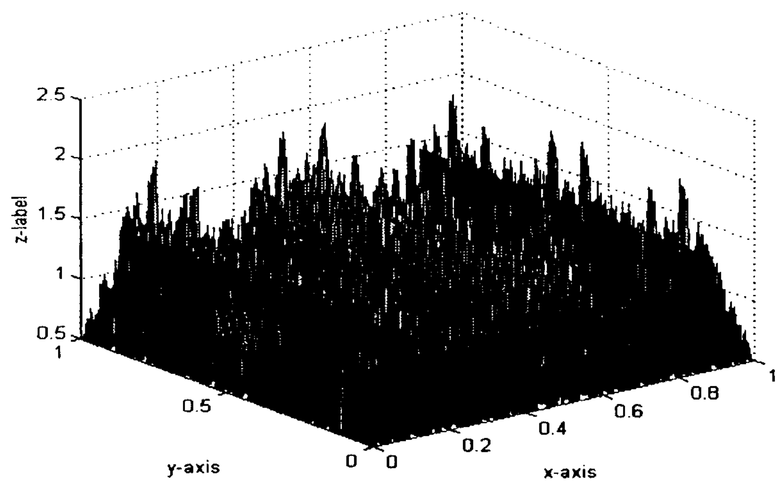

FIG. 2. Graph of $z = \lambda f(x) + \mu f(y) + e$ with $\lambda = 1.5$, $\mu = 2.5$, $e = 0.5$, $p_k = 3$, and $n_k = 2k$.

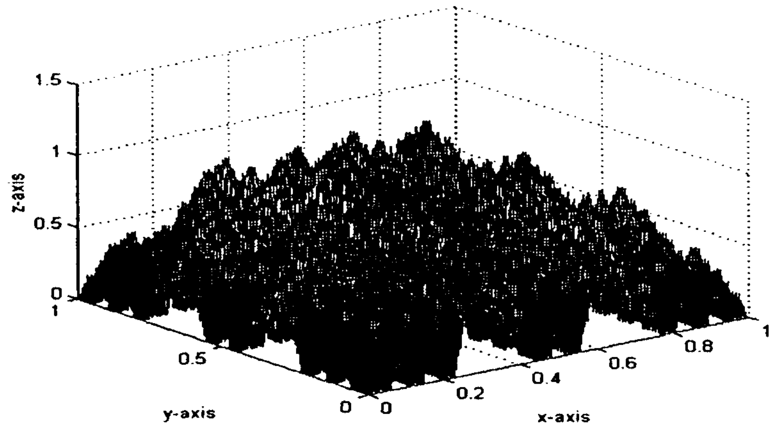

FIG. 3. Graph of $z = \lambda f(x) + \mu f(y) + e$ with $\lambda = 1$, $\mu = 1.5$, $e = 0$, $p_k = k + 2$, and $n_k = k$.

4. BOX-COUNTING AND PACKING DIMENSIONS OF THE ROUGH SURFACES

In this section we accurately calculate the box-counting and packing dimensions of the rough surface constructed in the last section.

For convenience, let G denote the rough surface, the graph of F being defined by Eq.(12); i.e.,

$$G = graph\, F = \{(x, y, F(x, y)) : (x, y) \in I^2\}.$$

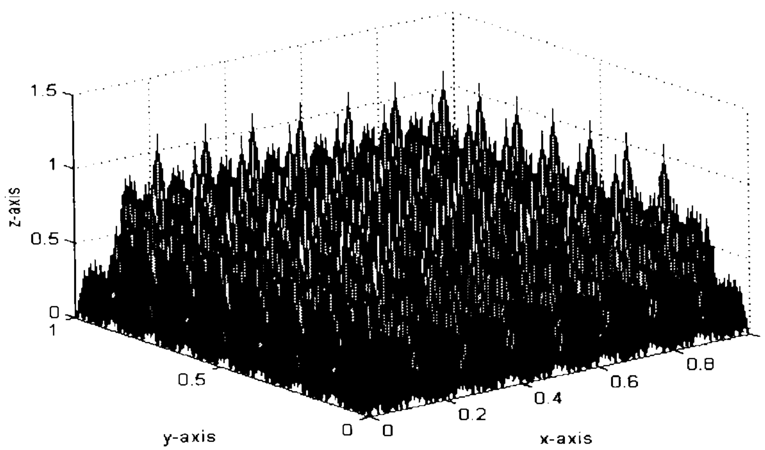

FIG. 4. Graph of $z = \lambda f(x) + \mu f(y) + e$ with $\lambda = 1.0$, $\mu = 1.5$, $e = 0$, $p_k = 8$, and $n_k = k$.

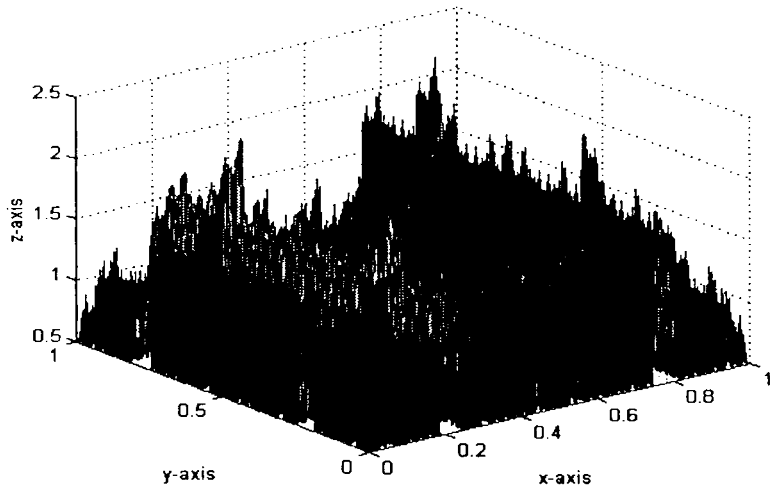

FIG. 5. Graph of $z = \lambda f(x) + \mu f(y) + e$ with $\lambda = 1.5$, $\mu = 2.5$, $e = 0.5$, $p_k = 2$, and $n_k = 2k$.

THEOREM 1. *Let* p_k, $k = 1, 2, 3, \ldots$, *be a bounded sequence of positive integers*; *and for some positive integer* m_0, *let* $n_k = m_0 k$, $k = 0, 1, 2, \ldots$. *Assume that one of the nondifferentiability conditions* (A) *and* (B) *holds*, *and*

$$\lim_{k\to\infty} \frac{k}{\log p_1 \cdots p_{n_k}} = c, \tag{13}$$

where c is a positive constant less than $1/\log 2$. *Then*, *the box-counting dimension of G is given by*

$$\dim_B G = 3 - c\log 2. \tag{14}$$

Proof. For a sufficiently large n_m, we divide each side of I^2 into $p_1 \cdots p_{n_m}$ subintervals of equal length, yielding $(p_1 \cdots p_{n_m})^2$ small squares Δ_{ij}, $i, j = 1, 2, \ldots, p_1 \cdots p_{n_m}$. Let $\delta_m = (p_1 \cdots p_{n_m})^{-1}$ and $N_{\delta_m}(G)$ be the number of the δ_m-mesh cubes which intersect G. Since the sequence $\{p_k\}$ is bounded and $\{n_k\} = \{m_0 k\}$, the decreasing sequence $\{\delta_m\}$ possesses the property that $\delta_{m+1} \geq c_0 \delta_m$, for some constant c_0, $0 < c_0 < 1$. In order to calculate the box-counting dimension of G, therefore, it suffices to take the limit through the discrete sequence $\{\delta_m\}$ in Eqs. (1) and (2) (for the reason see, e.g., [11]).

For any $P_{ij} = (x^{(ij)}, y^{(ij)}) \in \Delta_{ij}$, $Q_{ij} = (\hat{x}^{(ij)}, \hat{y}^{(ij)}) \in \Delta_{ij}$, let $x_k^{(ij)}$, $y_k^{(ij)}$, $\hat{x}_k^{(ij)}$, and $\hat{y}_k^{(ij)}$ be the corresponding kth digits of the Cantor series of $x^{(ij)}, y^{(ij)}, \hat{x}^{(ij)}$, and $\hat{y}^{(ij)}$, respectively. Obviously, for any $1 \leq k \leq n_m$, $x_k^{(ij)} = \hat{x}_k^{(ij)}$ and $y_k^{(ij)} = \hat{y}_k^{(ij)}$ hold.

The corresponding function value is, respectively,

$$f(x^{(ij)}) = 0.u_1^{(ij)} u_2^{(ij)} \cdots u_k^{(ij)} \cdots \quad (binary),$$
$$f(\hat{x}^{(ij)}) = 0.\hat{u}_1^{(ij)} \hat{u}_2^{(ij)} \cdots \hat{u}_k^{(ij)} \cdots \quad (binary),$$
$$f(y^{(ij)}) = 0.v_1^{(ij)} v_2^{(ij)} \cdots v_k^{(ij)} \cdots \quad (binary),$$

and

$$f(\hat{y}^{(ij)}) = 0.\hat{v}_1^{(ij)} \hat{v}_2^{(ij)} \cdots \hat{v}_k^{(ij)} \cdots \quad (binary).$$

Obviously, $u_k^{(ij)} = \hat{u}_k^{(ij)}$, $v_k^{(ij)} = \hat{v}_k^{(ij)}$, $1 \le k \le m$. Consequently,

$$|f(x^{(ij)}) - f(\hat{x}^{(ij)})| \le \sum_{k=m+1}^{\infty} \frac{|u_k^{(ij)} - \hat{u}_k^{(ij)}|}{2^k} \le \sum_{k=m+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^m}, \tag{15}$$

$$|f(y^{(ij)}) - f(\hat{y}^{(ij)})| \le \sum_{k=m+1}^{\infty} \frac{|v_k^{(ij)} - \hat{v}_k^{(ij)}|}{2^k} \le \sum_{k=m+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^m}. \tag{16}$$

Let $L = \sup\{|\lambda(x)|, |\mu(y)|, |\lambda'(x)|, |\mu'(y)| : 0 \le x, y \le 1\}$. From the differentiability of $\lambda(x)$, we have

$$\begin{aligned} |\lambda(x^{(ij)}) - \lambda(\hat{x}^{(ij)})| &\le L|x^{(ij)} - \hat{x}^{(ij)}| \le L \sum_{k=n_m+1}^{\infty} \frac{|x_k^{(ij)} - \hat{x}_k^{(ij)}|}{p_1 p_2 \cdots p_k} \\ &\le L \sum_{k=n_m+1}^{\infty} \frac{p_k - 1}{p_1 p_2 \cdots p_k} = L(p_1 p_2 \cdots p_{n_m})^{-1}. \end{aligned} \tag{17}$$

Similarly,

$$|\mu(y^{(ij)}) - \mu(\hat{y}^{(ij)})| \le L(p_1 p_2 \cdots p_{n_m})^{-1}. \tag{18}$$

Noting that, for any $x \in I$, $0 \le f(x) \le 1/2$ holds, by Eqs.(15)–(18) we obtain

$$\begin{aligned} |F(P_{ij}) - F(Q_{ij})| &\le |\lambda(x^{(ij)})||f(x^{(ij)}) - f(\hat{x}^{(ij)})| \\ &\quad + |\mu(y^{(ij)})||f(y^{(ij)}) - f(\hat{y}^{(ij)})| \\ &\quad + |f(\hat{x}^{(ij)})||\lambda(x^{(ij)}) - \lambda(\hat{x}^{(ij)})| \\ &\quad + |f(\hat{y}^{(ij)})||\mu(y^{(ij)}) - \mu(\hat{y}^{(ij)})| \\ &\le L[2^{-m+1} + (p_1 p_2 \cdots p_{n_m})^{-1}], \end{aligned}$$

which implies that

$$osc(F, \Delta_{ij}) = \sup_{P_{ij}, Q_{ij} \in \Delta_{ij}} |F(P_{ij}) - F(Q_{ij})| \le L[2^{-m+1} + (p_1 p_2 \cdots p_{n_m})^{-1}]$$

and

$$N_{\delta_m}(G) \le \sum_{i,j=1}^{p_1 \cdots p_{n_m}} \left[\frac{osc(F, \Delta_{ij})}{(p_1 \cdots p_{n_m})^{-1}} + 2 \right] \le (3L+2) \cdot 2^{-m} (p_1 \cdots p_{n_m})^3. \tag{19}$$

By hypothesis (13) and Eq.(19), we have

$$\lim_{m \to \infty} \frac{\log N_{\delta_m}(G)}{-\log (p_1 \cdots p_{n_m})^{-1}} \le 3 - c \log 2. \tag{20}$$

On the other hand, let $(x^{(ij)}, y^{(ij)})$ be a point lying on the left edge of Δ_{ij} and the Cantor series of $x^{(ij)}$,

$$x^{(ij)} = \sum_{k=1}^{n_m} \frac{x_k^{(ij)}}{p_1 \cdots p_k}, \qquad 0 \le x_k^{(ij)} \le p_k - 1.$$

We take

$$\bar{x}^{(ij)} = x^{(ij)} + \frac{\bar{x}_{n_m+1}^{(ij)}}{p_1 \cdots p_{n_m} p_{n_m+1}}, \qquad 0 < \bar{x}_{n_m+1}^{(ij)} < p_{n_m+1} - 1, \tag{21}$$

whose corresponding function value then is given by

$$f(\bar{x}^{(ij)}) = 0.u_1 u_2 \cdots u_m u_{m+1} u_{m+2} \cdots \ (binary).$$

It is easy to see from Eq. (21) that

$$u_{m+2} = 1 - u_{m+1} \qquad \text{and} \qquad u_k = u_{m+2} \ (k \ge m+3). \tag{22}$$

Take again

$$\tilde{x}^{(ij)} = x^{(ij)} + \sum_{k=n_m+1}^{n_{m+2}} \frac{p_k - 1}{p_1 \cdots p_k}. \tag{23}$$

The corresponding function value is given by

$$f(\tilde{x}^{(ij)}) = 0.v_1 v_2 \cdots v_m v_{m+1} v_{m+2} \cdots \ (binary).$$

From the choice of $\tilde{x}^{(ij)}$, we have that

$$v_k = u_k, 1 \le k \le m, \text{ and } v_{m+2} = v_{m+1}, \ v_{m+3} = 1 - v_{m+2}. \tag{24}$$

From Eqs. (22) and (24), it is easy to see that if $v_{m+1} = 1 - u_{m+1}$, then $v_{m+2} = u_{m+2}$, and if $v_{m+1} = u_{m+1}$, then $v_{m+2} = 1 - u_{m+2}$ and $v_{m+3} = u_{m+3}$. So in either case we always have

$$|f(\bar{x}^{(ij)}) - f(\tilde{x}^{(ij)})| = \left| \sum_{k=m+1}^{\infty} \frac{u_k - v_k}{2^k} \right| \ge 2^{-(m+3)}. \tag{25}$$

Hence, for any Δ_{ij}, we can choose two points $(\bar{x}^{(ij)}, y^{(ij)})$ and $(\tilde{x}^{(ij)}, y^{(ij)})$ in Δ_{ij} such that

$$|F(\bar{x}^{(ij)}, y^{(ij)}) - F(\tilde{x}^{(ij)}, y^{(ij)})| \\ \geq \|\lambda(\bar{x}^{(ij)})\| f(\bar{x}^{(ij)}) - f(\tilde{x}^{(ij)})| - |f(\tilde{x}^{(ij)})\| \lambda(\bar{x}^{(ij)}) - \lambda(\tilde{x}^{(ij)})\|. \quad (26)$$

We consider the term in absolute value on the right-hand side of the inequality (26). Noting that $|\bar{x}^{(ij)} - \tilde{x}^{(ij)}| < (p_1 \cdots p_{n_m})^{-1}$ and inequality (25), we have

$$\begin{aligned} &|\lambda(\bar{x}^{(ij)})||f(\bar{x}^{(ij)}) - f(\tilde{x}^{(ij)})| - |f(\tilde{x}^{(ij)})||\lambda(\bar{x}^{(ij)}) - \lambda(\tilde{x}^{(ij)})| \\ &\geq \lambda_1 2^{-(m+3)} - 2^{-1} L(p_1 \cdots P_{n_m})^{-1} \\ &= 2^{-1}(p_1 \cdots p_{n_m})^{-1}(2^{-2}\lambda_1 \cdot 2^{-m} p_1 \cdots p_{n_m} - L), \end{aligned} \quad (27)$$

where $\lambda_1 = \min\{|\lambda(x)| : 0 \leq x \leq 1\} > 0$. By the conditions of nondifferentiability, we have $L < 2^{-3}\lambda_1 \cdot 2^{-m} p_1 \cdots p_{n_m}$, provided that m is taken sufficiently large. Hence, it follows from Eqs. (26) and (27) that

$$|F(\bar{x}^{(ij)}, y^{(ij)}) - F(\tilde{x}^{(ij)}, y^{(ij)})| \geq 2^{-4}\lambda_1 \cdot 2^{-m},$$

which implies that

$$osc(F, \Delta_{ij}) \geq 2^{-4}\lambda_1 \cdot 2^{-m}.$$

Therefore

$$N_{\delta_m}(G) \geq \sum_{i,j=1}^{p_1 \cdots p_{n_m}} \left[\frac{osc(F, \Delta_{ij})}{(p_1 \cdots p_{n_m})^{-1}}\right] \geq 2^{-4}\lambda_1 \cdot 2^{-m}(p_1 \cdots p_{n_m})^3,$$

and consequently

$$\lim_{m\to\infty} \frac{\log N_{\delta_m}(G)}{-\log (p_1 \cdots p_{n_m})^{-1}} \geq 3 - c\log 2. \quad (28)$$

Combining Eqs. (20) and (28) thus yields Eq. (14). This finishes the proof of Theorem 1.

The following theorem gives the result of the packing dimension of G which reveals the relation between the box-counting dimension and the packing dimension of G.

THEOREM 2. *In the setting of Theorem* 1, *the packing and box-counting dimensions of G are the same*; *i.e.*,

$$\dim_P G = \dim_B G = 3 - c\log 2. \quad (29)$$

Proof. Let $V \subset R^3$ be any open set such that $V \cap G \neq \emptyset$. For the compact set G, we show that G satisfies Eq. (3).

Through properly choosing index n_{r_0}, we can divide I^2 into $(p_1 \cdots p_{n_{r_0}})^2$ small squares of equal edge length, say Δ_{ij}, $i, j = 1, 2, \ldots, p_1 \cdots p_{n_{r_0}}$, such that the surface segment $G_{i_0 j_0}$ over some small square $\Delta_{i_0 j_0}$, $i_0, j_0 \in \{1, 2, \ldots, p_1 \cdots p_{n_{r_0}}\}$, is included in $V \cap G$.

Now we consider the upper box-counting dimension of $G_{i_0 j_0}$. For a sufficiently large $n_s > n_{r_0}$, we divide $\Delta_{i_0 j_0}$ into $(p_{n_{r_0}+1} \cdots p_{n_s})^2$ smaller squares of equal edge length, say σ_{ij}, $i, j = 1, 2, \ldots, p_{n_{r_0}+1} \cdots p_{n_s}$. The edge length of each σ_{ij} is $(p_1 \cdots p_{n_s})^{-1}$. As in the proof of Theorem 1, we can choose two points $(\bar{x}^{(ij)}, y^{(ij)})$, $(\tilde{x}^{(ij)}, y^{(ij)})$ in each σ_{ij} such that

$$|f(\bar{x}^{(ij)}) - f(\tilde{x}^{(ij)})| \geq 2^{-(s+3)},$$

and therefore

$$osc(F, \sigma_{ij}) \geq 2^{-4} \lambda_1 \cdot 2^{-s}.$$

Consequently,

$$\begin{aligned} N_{\delta_s}(G_{i_0 j_0}) &\geq \sum_{i,j=1}^{p_{n_{r_0}+1} \cdots p_{n_s}} \left[\frac{2^{-4} \lambda_1 2^{-s}}{(p_1 \cdots p_{n_s})^{-1}} \right] \\ &= 2^{-4} \lambda_1 (p_1 \cdots p_{n_{r_0}})^{-2} \cdot 2^{-s} (p_1 \cdots p_{n_s})^3 \end{aligned}$$

and

$$\overline{\lim_{s \to \infty}} \frac{\log N_{\delta_s}(G_{i_0 j_0})}{-\log (p_1 \cdots p_{n_s})^{-1}} \geq 3 - c \log 2. \tag{30}$$

By the monotonicity of upper box-counting dimension and Eq. (30), we thus have

$$\overline{\dim}_B(V \cap G) \geq \overline{\dim}_B G_{i_0 j_0} \geq 3 - c \log 2.$$

Since $\overline{\dim}_B(V \cap G) \leq \overline{\dim}_B G = \dim_B G = 3 - c \log 2$, it follows that $\overline{\dim}_B(V \cap G) = \dim_B G$. That is, Eq. (3) is satisfied with $F = G$. By Lemma 1, Eq. (29) then follows. This finishes the proof of Theorem 2.

5. ESTIMATE OF THE HAUSDORFF DIMENSION

In this section we discuss the Hausdorff dimension of the rough surface G. In the case when $p_k = b$ (here $b \geq 3$ is an integer), $k = 1, 2, 3, \ldots$, (hence the Cantor series of x degenerates to the b-adic fractional expression), we give the following estimate for the Hausdorff dimension of G.

THEOREM 3. *Let $p_k = b$ ($b \geq 3$ is an integer), $k = 1, 2, 3, \ldots, n_k = m_0 k$, $k = 0, 1, 2, \ldots$, where m_0 is a positive integer. Then*

$$2 + (1 - m_0 \log_2 b) \log_M (1 - M^{-2}) < \dim_H G \leq 3 - {m_0}^{-1} \log_b 2, \qquad (31)$$

where $M = [2/c_1] + 1$ and $[x]$ denotes the integer part of $x > 0$; $c_1 > 0$ may be taken to be any constant less than $[\lambda_1/(2^{7+1/m_0} 3L)]^{1/\alpha}$, where $\alpha = \frac{1}{m_0} \log_b 2$.

Proof. Since $\dim_H G \leq \dim_B G$, we can deduce immediately the inequality on the right-hand side of Eq. (31) from the hypotheses of Theorem 3 and the result of Theorem 1. We therefore only need to justify the inequality on the left-hand side of Eq. (31).

In order to do this, we need to show that for the rough surface $G \subset R^3$ the two conditions of Lemma 2 are satisfied by G.

With reference to Lemma 2, let $W_1 = R^2, W_2 = R$, and $K = G$, then $m_{W_1}(\pi_{W_1}(G)) = 1 > 0$. Suppose that $D \subset I^2$ is any canonical square with a sufficiently small edge length r, then there exists a positive integer n_p such that

$$\frac{1}{b^{n_p+1}} \leq r < \frac{1}{b^{n_p}}. \qquad (32)$$

We divide each side of I^2 into $b^{n_{p+2}}$ subintervals of equal length, yielding $b^{2n_{p+2}}$ small squares, hence there exists a $b^{-n_{p+2}}$-mesh square D_1 such that D_1 is included in D. For arbitrary $(x, y), (x', y') \in D_1$, by applying techniques similar to those used in the preceding section we obtain

$$\sup_{D_1} |F(x, y) - F(x', y')| \geq 2^{-6} \lambda_1 \cdot 2^{-p}. \qquad (33)$$

Let $\alpha = (1/m_0) \log_b 2$. Clearly, $0 < \alpha < 1$ since $b \geq 3$ and $m_0 \geq 1$. Noting Eqs. (32) and (33), we have

$$\sup_{D_1} |F(x, y) - F(x', y')| \geq 2^{-6} \lambda_1 \left(\frac{1}{b^{n_p}} \right)^{\alpha} > 2^{-6} \lambda_1 \cdot r^{\alpha}. \qquad (34)$$

From Eq. (34), we can assert that for any $z \in R$ there must exist a point $(x_0, y_0) \in D_1$ such that

$$|F(x_0, y_0) - z| \geq 2^{-7} \lambda_1 \cdot r^{\alpha}. \qquad (35)$$

In fact, if there exists some $z_0 \in R$ such that for any $(x, y) \in D_1$, $|F(x, y) - z_0| < 2^{-7} \lambda_1 \cdot r^{\alpha}$ holds, then for arbitrary $(x, y), (x', y') \in D_1$, $|F(x, y) - F(x', y')| < 2^{-6} \lambda_1 \cdot r^{\alpha}$ follows, which contradicts Eq. (34).

Now we proceed to look for a canonical square $D' \subset D_1$, with center at (x_0, y_0) and edge length d such that for every $(x, y) \in D'$,

$$|F(x, y) - z| > \frac{1}{2} c_2 r^{\alpha}$$

holds, where $c_2 > 0$ is some sufficiently small constant. The magnitude of d first should be determined.

For arbitrary $(x, y), (x', y') \in D'$, there exists a positive integer n_q such that

$$\frac{1}{b^{n_q+1}} \leq |x - x'| < \frac{1}{b^{n_q}} \qquad \text{and} \qquad |y - y'| < \frac{1}{b^{n_q}} \tag{36}$$

or

$$\frac{1}{b^{n_q+1}} \leq |y - y'| < \frac{1}{b^{n_q}} \qquad \text{and} \qquad |x - x'| < \frac{1}{b^{n_q}}. \tag{37}$$

This means that

$$|f(x) - f(x')| \leq \frac{1}{2^q} \qquad \text{and} \qquad |f(y) - f(y')| \leq \frac{1}{2^q}.$$

Consequently,

$$\begin{aligned} |F(x, y) - F(x', y')| &\leq L\left(2^{-q+1} + \frac{1}{b^{n_q}}\right) < 3L \cdot 2^{-q} \\ &\leq 3L2^{\frac{1}{m_0}} \cdot \delta^\alpha \leq 3L2^{\frac{1}{m_0}} \cdot d^\alpha, \end{aligned} \tag{38}$$

where $\delta = \max\{|x - x'|, |y - y'|\}$.

Let

$$3L2^{\frac{1}{m_0}} d^\alpha < 2^{-7}\lambda_1 r^\alpha - \frac{1}{2} c_2 r^\alpha. \tag{39}$$

To make the inequality (39) hold, it is enough to take an arbitrary d satisfying $3L2^{1/m_0} d^\alpha < 2^{-7}\lambda_1 r^\alpha$, provided that $c_2 > 0$ is taken sufficiently small. Hence we choose

$$d < \left[\frac{\lambda_1}{2^{7+\frac{1}{m_0}} 3L}\right]^{\frac{1}{\alpha}} r.$$

For any $(x, y) \in D'$, by Eqs. (35), (38), and (39) it then follows that

$$\begin{aligned} |F(x, y) - z| &\geq |F(x_0, y_0) - z| - |F(x, y) - F(x_0, y_0)| \\ &> 2^{-7}\lambda_1 r^\alpha - \left(2^{-7}\lambda_1 r^\alpha - \frac{1}{2} c_2 r^\alpha\right) = \frac{1}{2} c_2 r^\alpha. \end{aligned}$$

Therefore we conclude that $\{D' \times D(z, c_2 r^\alpha)\} \cap G = \emptyset$. From Lemma 2 and Remark 1, the inequality on the left-hand side of (31) is derived.

Remark 2. Under certain conditions that are slightly weaker than those of Theorem 3, we can obtain the Hölder continuity of the function $F(x, y)$, that is, if $2 \leq p_k \leq b$, $k = 1, 2, \ldots$, $\{n_k\} = \{m_0 k\}$, m_0 is a positive integer, and $\alpha = (1/m_0)\log_b 2$, then the function $F(x, y)$ is of class Lip α.

Remark 3. In this paper we have constructed a class of rough surfaces by means of a previously defined function $f(x)$ and have discussed their fractal dimensions, which are an indicator of their roughness. Analogously, applying other construction techniques, such as considering $F(x, y) = \lambda f(x) f(y)$, where λ is a nonzero constant, we are able to obtain other types of rough surfaces. In addition, it is easy to see that our construction can be generalized quite easily to n dimensions with only minor changes in the proofs. We believe that these surfaces with simple constructions but complicated structures will have actual applicability in computer graphics.

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